Convergence of approximate deconvolution models to the filtered Navier-Stokes Equations

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Abstract

We consider a 3D Approximate Deconvolution Model (ADM) which belongs to the class of Large Eddy Simulation (LES) models. We work with periodic boundary conditions and the filter that is used to average the fluid equations is the Helmholtz one. We prove existence and uniqueness of what we call a "regular weak" solution $(\mathbf{w}_N, q_N)$ to the model, for any fixed order $N \in \mathbb{N}$ of deconvolution. Then, we prove that the sequence $\{(\mathbf{w}_N, q_N)\}_{N\in\mathbb{N}}$ converges –in some sense– to a solution of the filtered Navier-Stokes equations, as N goes to infinity. This rigorously shows that the class of ADM models we consider have the most meaningful approximation property for averages of solutions of the Navier-Stokes equations.

MCS Classification : 76D05, 35Q30, 76F65, 76D03

Key-words : Navier-Stokes equations, Large eddy simulation, Deconvolution models.

1 Introduction

It is well-known that the Kolmogorov theory predicts that simulating turbulent flows by using the Navier-Stokes Equations

$$
\begin{aligned}
\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}, \\
\nabla \cdot \mathbf{u} &= 0, \\
\mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}).
\end{aligned}
\tag{1.1}
$$

requires $\mathcal{N} = O(Re^{9/4})$ degrees of freedom, where $Re = UL\,\nu^{-1}$ denotes the Reynolds number and U and L are a typical velocity and length, respectively. This number $\mathcal{N}$ is too large, in comparison with memory capacities of actual computers, to perform a Direct Numerical Simulation (DNS). Indeed, for realistic flows, such as for instance geophysical flows, the Reynolds number is order 10^8, yielding $\mathcal{N}$ of order 10^{18}.... This is why one aims at computing at least the "mean values" of the flows fields, which are the velocity field $\mathbf{u} = (u^1, u^2, u^3)$ and the scalar pressure field p. This is heuristically motivated from the fact that some gross characteristics of the flow behave in a more orderly manner. In the spirit of the work started probably with Reynolds, this correspond in finding a suitable computational decomposition

$$
\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}' \qquad \text{and} \qquad p = \overline{p} + p',
$$

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where the primed variables are fluctuations around the over-lined mean fields. Fluctuations can be disregarded since generally in applications knowledge of the mean flow is enough to extract relevant information on the fluid motion.

The "mean values" can be defined in several ways (time or space average, statistical averages...); in particular, if one denotes the means fields by $\overline{\mathbf{u}}$ and $\overline{p}$, and by assuming that the averaging operation commutes with differential operators, one gets the filtered Navier-Stokes equations

$$\begin{aligned} \partial_t \overline{\mathbf{u}} + \nabla \cdot (\overline{\mathbf{u} \otimes \mathbf{u}}) - \nu \Delta \overline{\mathbf{u}} + \nabla \overline{p} &= \overline{\mathbf{f}}, \\ \nabla \cdot \overline{\mathbf{u}} &= 0, \\ \overline{\mathbf{u}}(0, \mathbf{x}) &= \overline{\mathbf{u}_0}(\mathbf{x}). \end{aligned} \tag{1.2}$$

This raises the question of the *interior closure problem*, that is the modeling of the tensor $R(\mathbf{u}) = \overline{\mathbf{u} \otimes \mathbf{u}}$ in terms of the filtered variables $(\overline{\mathbf{u}}, \overline{p})$. Classical Large Eddy Simulations (LES) models approximate R by $\mathbf{w} \otimes \mathbf{w} - \nu_T(\mathbf{k}/\mathbf{k}_c)\mathcal{D}(\mathbf{w})$ where $\mathbf{w} \approx \overline{\mathbf{u}}$, and $\mathcal{D}(\mathbf{w}) = (1/2)(\nabla \mathbf{w} + \nabla \mathbf{w}^T)$. Here ν_T is an eddy viscosity based on a "cut frequency" $\mathbf{k}_c$ (see a general setting in [21]). We introduce the new variable $\mathbf{w}$ since when using any approximation for $R(\mathbf{u})$, one does not write the differential equation satisfied by $\overline{\mathbf{u}}$, but an equation satisfied by another field $\mathbf{w}$ which is *hopefully* close enough to $\mathbf{u}$.

Another way, that avoids eddy viscosities, consists in approaching R by a quadratic term of the form $B(\mathbf{w}, \mathbf{w})$. J. Leray [16] already used in the 1930s the approximation (with our LES notation) $B(\mathbf{w}, \mathbf{w}) = \overline{\mathbf{w}} \otimes \mathbf{w}$ to get smooth approximations to the Navier-Stokes Equations. This approximations yield the recent Leray-alpha fashion models, considered to be LES models, and a broad class of close models (see *e.g.* [4, 8, 2, 10, 17]). In [12, 13], we also have studied the model defined by $B(\mathbf{w}, \mathbf{w}) = \overline{\mathbf{w} \otimes \mathbf{w}}$, which has also strict connections with scale-similarity models.

The model we study in this paper, is the Approximate Deconvolution Model (ADM), first introduced by Adams and Stolz [22, 1], so far as we know. This model is defined by $B(\mathbf{w}, \mathbf{w}) = \overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})}$. Here the operator G is defined thanks to the Helmholtz filter (cf. (2.2)-(2.4) below) by $G(\mathbf{v}) = \overline{\mathbf{v}}$, where in the paper $G = (\mathrm{I} - \alpha^2 \Delta)^{-1}$, and the operator D_N has the form $D_N = \sum_{n=0}^{N} (\mathrm{I} - G)^n$. Therefore, the initial value problem we consider is:

$$\begin{aligned} \partial_t \mathbf{w} + \nabla \cdot (\overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})}) - \nu \Delta \mathbf{w} + \nabla q &= \overline{\mathbf{f}}, \\ \nabla \cdot \mathbf{w} &= 0, \\ \mathbf{w}(0, \mathbf{x}) &= \overline{\mathbf{u}_0}(\mathbf{x}), \end{aligned} \tag{1.3}$$

and we are working with periodic boundary conditions. We already observed that the equations (1.3) are not the equations (1.2) satisfied by $\overline{\mathbf{u}}$, but we are aimed at considering (1.3) as an approximation of (1.2). This is mathematically sound since formally $D_N \to A := \mathrm{I} - \alpha^2 \Delta$, in the limit $N \to +\infty$, hence again formally (1.3) will become the filtered Navier-Stokes (1.4). What is more challenging is to understand whether this property is true or not, in the sense that one would like to show that as the approximation parameter N grows, then

$$\mathbf{w} \to \overline{\mathbf{u}} \qquad \text{and} \qquad q \to \overline{p}.$$

One would like to prove that solutions of the model converge to averages of the true quantities, since, we recall that the main goal of LES as a computational tool is to approximate the averages of the flow, which are the only interesting and computable quantities. To this end we want to point out that, beside the technical mathematical difficulties, proving

results of approximation of single trajectories

$$\mathbf{w} \to \mathbf{u} \qquad \text{and} \qquad q \to p,$$

is not really in the "rules of the game," because the consistency of the model towards single (generally strong or not computable) solutions to the Navier-Stokes equations is not the most interesting point. Anyway, such convergence is generally known only for a few models, as for instance for many of the alpha-models, as $\alpha \to 0^+$. To support again this point of view, observe that generally $\alpha > 0$ is related to the smallest resolved/resolvable scale, hence $\alpha = \mathcal{O}(h)$, where h is the mesh size. Reducing α will mean resolving completely the flow, hence performing a DNS instead of a LES.

To our knowledge such a "*well posedness*," i.e. proving that $\mathbf{w} \to \overline{\mathbf{u}}$, is not known for any LES model: To our knowledge there are no results showing (not only formally, but also rigorously) that the solution of a LES model $(\mathbf{w}, q)$ is close or converges in some sense to the *averages* $(\overline{\mathbf{u}}, \overline{p})$.

To continue the introduction to our new results, we recall that model (1.3) has already been considered in [14] where we studied the residual stress. It has also been studied in Dunca and Epshteyn [7], where it has been proved the existence of a unique "smooth enough" solution for periodic boundary conditions. In [7] it is also shown that, that the sequence of models (1.3) goes -in a certain meaning- to the Navier-Stokes Equations when $\alpha \to 0^+$, for N fixed. Notice that the model intensively investigated in [12, 13, 3], where $B(\mathbf{w}, \mathbf{w}) = \overline{\mathbf{w} \otimes \mathbf{w}}$, is the special case $N = 0$ and it is also called simplified-Bardina, since it resembles some features of the scale similarity models.

The main topic of this paper is then to study what happens when N goes to infinity in (1.3). We prove that the sequence of models (1.3) converges, in some sense, to the averaged Navier-Stokes equations (1.2), when the typical scale of filtration (called $\alpha > 0$) remains fixed and the boundary conditions are the periodic ones. Before analyzing such convergence we need to prove existence of smooth enough solutions. To this end we needed to completely revisit the approach in [7]. To be more precise, let $\mathbb{T}^3$ be the $3D$ torus and let $(\mathbf{w}_N, q_N)$, with

$$\begin{aligned}
&\mathbf{w}_N \in L^2([0,T]; H^2(\mathbb{T}_3)^3) \cap L^\infty([0,T]; H^1(\mathbb{T}_3)^3),\\
&q_N \in L^2([0,T]; W^{1,2}(\mathbb{T}_3) \cap L^{5/3}([0,T]; W^{2,5/3}(\mathbb{T}_3)),
\end{aligned}$$

denote the solution of the ADM model (1.3), where T is a fixed time, that can eventually be taken to be equal to ∞, assuming $\mathbf{f} \in L^2([0,T]; (H^1(\mathbb{T}_3)^3)')$ and $\mathbf{u}_0 \in L^2(\mathbb{T}_3)^3$, an assumption that we do throughout the paper. We are able to prove existence (cf. Theorem 3.1) in such class and our main result is the following.

Theorem 1.1. *From the sequence $\{(\mathbf{w}_N, q_N)\}_{N\in\mathbb{N}}$ one can extract a sub-sequence (still denoted $\{(\mathbf{w}_N, q_N)\}_{N\in\mathbb{N}}$) such that*

$$\begin{aligned}
&\mathbf{w}_N \to \mathbf{w} && \textit{weakly in } L^2([0,T]; H^2(\mathbb{T}_3)^3) \cap L^\infty([0,T]; H^1(\mathbb{T}_3)^3),\\
&\mathbf{w}_N \to \mathbf{w} && \textit{strongly in } L^p([0,T]; H^1(\mathbb{T}_3)^3), \quad \forall\, 1 \le p < +\infty,\\
&q_N \to q && \textit{weakly in } L^2([0,T]; W^{1,2}(\mathbb{T}_3) \cap L^{5/3}([0,T]; W^{2,5/3}(\mathbb{T}_3)),
\end{aligned}$$

and such that the system

$$\begin{aligned}
\partial_t \mathbf{w} + \nabla \cdot (\overline{A\mathbf{w} \otimes A\mathbf{w}}) - \nu \Delta \mathbf{w} + \nabla q &= \overline{\mathbf{f}},\\
\nabla \cdot \mathbf{w} &= 0,\\
\mathbf{w}(0, \mathbf{x}) &= \overline{\mathbf{u}_0}(\mathbf{x}),
\end{aligned} \tag{1.4}$$

holds in the distributional sense, where we recall that $A = G^{-1} = I - \alpha^2\Delta$*. Moreover, the following energy inequality holds:*

$$\frac{1}{2}\frac{d}{dt}\|A\mathbf{w}\|^2 + \nu\|\nabla A\mathbf{w}\|^2 \leq \langle \mathbf{f}, A\mathbf{w}\rangle. \tag{1.5}$$

As a consequence of Theorem 1.1, we deduce that the field $(\mathbf{u}, p) = (A\mathbf{w}, Aq)$ is a dissipative (Leray-Hopf) solution to the Navier-Stokes Equations (1.1).

Remark 1.1. *If one rewrites system* (1.2) *in terms of the variables* $\mathbf{w} = \overline{\mathbf{u}}$ *and* $\mathbf{u} = A\overline{\mathbf{u}} = A\mathbf{w}$*, one obtains exactly the system* (1.4)*. This is not a LES model, since it is just a change of variables. The LES modeling comes into the equations with the approximation of the operator* A *by means of the family* $\{D_N\}_{N\in\mathbb{N}}$*.*

Plan of the paper. Since the paper deals mainly with the mathematical properties of the model, we start in Section 2 by giving a precise definition of our filter through the Helmholtz equation and we sketch a reminder of the basic properties of the deconvolution operator D_N. The precise knowledge of the filter is one of the critical points in the analysis we will perform. We also claim that, beside some basic knowledge of functional analysis, we have been able to simplify the proof in order to employ just the classical energy and compactness methods. Roughly speaking, we needed to find the correct *multipliers* and –at least in principle– the proof of the main result should be readable also from practitioners. Then, we show in Section 3 an existence and uniqueness result for system (1.3). Even if this result has already been obtained by Dunca and Epshteyn [7], our proof is shorter and uses different arguments, useful for proving our main convergence result. Indeed, Dunca and Epshteyn proved initially a smart but very technical formula about D_N in terms of series of $(-\Delta)^k$, but they did not get uniform estimates in N. This is why their proof cannot help for passing to the limit when N goes to infinity. Our first main observation in this paper is that one can get very easily an estimate uniform in N for $A^{1/2}D_N^{1/2}(\mathbf{w})$, that also yields estimates for $D_N(\mathbf{w})$ and $\mathbf{w}$ itself, always uniform in N. The *leitmotiv* of the paper is to prove estimates *independent* of N.
Finally, Section 4 is devoted to the proof of Theorem 1.1. To prove it, we use the estimates, uniform in N, obtained in the proof of the existence result. Another main ingredient of the proof, is the derivation of an estimate for $\partial_t D_N(\mathbf{w}_N)$. Good estimates for this term yield a compactness property (à la Aubin-Lions) for $D_N(\mathbf{w}_N)$, which allows us to pass to the limit in the non linear term. We also note that in our argument we keep control of the pressure, since it is needed in some arguments and we do not simply neglect it by projecting the equations over divergence-free vector fields.

Acknowledgements. The work of Roger Lewandowski is partially supported by the ANR project 08FA300-01. Roger Lewandowski also warmly thanks the department of applied math. of the University of Pisa for the hospitality, where part of the work has been done. Luigi C. Berselli gratefully acknowledges the hospitality of IRMAR, Université Rennes 1, where part of the work has been done.

2 General Background

2.1 Orientation

This section is devoted first to the definition of the function spaces that we use, next to the definition of the filter through the Helmholtz equation, and finally to what we call the "deconvolution operator." There is nothing new here that is not already proved in former

papers. This is why we restrict our-selves to what we need for our display and we skip out proofs and technical details. Those details can be proved by standard analysis and the reader can check them in several references already quoted in the introduction and also quoted below in the text.

2.2 Function spaces

In the sequel we will use the customary Lebesgue L^p and Sobolev $W^{k,p}$ and $W^{s,2} = H^s$ spaces. Since we work with periodic boundary conditions we can better characterize the divergence-free spaces we need. In fact, the spaces we consider are well-defined by using Fourier Series on the 3D torus $\mathbb{T}_3$ defined just below. Let be given $L \in \mathbb{R}_+^\star = \{x \in \mathbb{R} : x > 0\}$, and define $\Omega := [0, L]^3 \subset \mathbb{R}^3$. We denote by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the orthonormal basis of $\mathbb{R}^3$, and by $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ the standard point in $\mathbb{R}^3$. We put $\mathcal{T}_3 := 2\pi\mathbb{Z}^3/L$. Let $\mathbb{T}_3$ be the torus defined by $\mathbb{T}_3 = \left(\mathbb{R}^3/\mathcal{T}_3\right)$. We use $\|\cdot\|$ to denote the $L^2(\mathbb{T}_3)$ norm and associated operator norms. We always impose the zero mean condition $\displaystyle\int_\Omega \phi \, d\mathbf{x} = 0$ on $\phi = \mathbf{w}, p, \mathbf{f}$, or $\mathbf{w}_0$. We define, for a general exponent $s \geq 0$,

$$\mathbf{H}_s = \left\{ \mathbf{w} : \mathbb{T}_3 \to \mathbb{R}^3, \ \mathbf{w} \in H^s(\mathbb{T}_3)^3, \quad \nabla \cdot \mathbf{w} = 0, \quad \int_{\mathbb{T}_3} \mathbf{w} \, d\mathbf{x} = \mathbf{0} \right\},$$

where $H^s(\mathbb{T}_3)^k = \left[H^s(\mathbb{T}_3)\right]^k$, for all $k \in \mathbb{N}$ (If $0 \leq s < 1$ the condition $\nabla \cdot \mathbf{w} = 0$ must be understood in a weak sense).
For $\mathbf{w} \in \mathbf{H}_s$, we can expand the velocity field in a Fourier series

$$\mathbf{w}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{T}_3^\star} \widehat{\mathbf{w}}_{\mathbf{k}} e^{+i\mathbf{k}\cdot\mathbf{x}}, \text{ where } \mathbf{k} \in \mathcal{T}_3^\star \text{ is the wave-number},$$

and the Fourier coefficients are given by

$$\widehat{\mathbf{w}}_{\mathbf{k}} = \frac{1}{|\mathbb{T}_3|} \int_{\mathbb{T}_3} \mathbf{w}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}.$$

The magnitude of $\mathbf{k}$ is defined by

$$k := |\mathbf{k}| = \{|k_1|^2 + |k_2|^2 + |k_3|\}^{\frac{1}{2}}.$$

We define the $\mathbf{H}_s$ norms by

$$\|\mathbf{w}\|_s^2 = \sum_{\mathbf{k} \in \mathcal{T}_3^\star} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}_{\mathbf{k}}|^2,$$

where of course $\|\mathbf{w}\|_0^2 = \|\mathbf{w}\|^2$. The inner products associated to these norms are

$$(\mathbf{w}, \mathbf{v})_{\mathbf{H}_s} = \sum_{\mathbf{k} \in \mathcal{T}_3^\star} |\mathbf{k}|^{2s} \widehat{\mathbf{w}}_{\mathbf{k}} \cdot \overline{\widehat{\mathbf{v}}_{\mathbf{k}}}, \tag{2.1}$$

where here, and without risk of confusion with the filter defined later, $\overline{\widehat{\mathbf{v}}_{\mathbf{k}}}$ denotes the complex conjugate of $\widehat{\mathbf{v}}_{\mathbf{k}}$. This means that if $\widehat{\mathbf{v}}_{\mathbf{k}} = (v_{\mathbf{k}}^1, v_{\mathbf{k}}^2, v_{\mathbf{k}}^3)$, then $\overline{\widehat{\mathbf{v}}_{\mathbf{k}}} = (\overline{v_{\mathbf{k}}^1}, \overline{v_{\mathbf{k}}^2}, \overline{v_{\mathbf{k}}^3})$. Since we are looking for real valued vectors fields, we have the natural relation, for any field denoted by $\mathbf{w}$:

$$\widehat{\mathbf{w}}_{\mathbf{k}} = \overline{\widehat{\mathbf{w}}_{-\mathbf{k}}}, \qquad \forall \, \mathbf{k} \in \mathcal{T}_3^\star.$$

Therefore, our space $\mathbf{H}_s$ is a closed subset of the space $\mathbb{H}_s$ of complex valued functions

$$\mathbb{H}_s = \left\{ \mathbf{w} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{w}}_{\mathbf{k}} e^{+i\mathbf{k}\cdot\mathbf{x}} : \quad \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}_{\mathbf{k}}|^2 < \infty, \ \mathbf{k}\cdot\widehat{\mathbf{w}}_{\mathbf{k}} = 0 \right\},$$

equipped with the Hilbert structure given by (2.1). It can be shown (see e.g. [6]) that when s is an integer, $\|\mathbf{w}\|_s^2 = \|\nabla^s \mathbf{w}\|^2$. One also can prove that for general $s \in \mathbb{R}$, $(\mathbb{H}_s)' = \mathbb{H}_{-s}$ (see in [19]).

2.3 About the Filter

We now recall the main properties of the Helmholtz filter. In the sequel $\alpha > 0$, denotes a given fixed number and $\mathbf{w} \in \mathbf{H}_s$. We consider the Stokes-like problem for $s \geq -1$:

$$\begin{aligned} -\alpha^2 \Delta \overline{\mathbf{w}} + \overline{\mathbf{w}} + \nabla \pi &= \mathbf{w} \qquad \text{in } \mathbb{T}_3, \\ \nabla \cdot \overline{\mathbf{w}} &= 0 \qquad \text{in } \mathbb{T}_3, \end{aligned} \tag{2.2}$$

and in addition, $\displaystyle\int_{\mathbb{T}_3} \pi \, d\mathbf{x} = 0$ to have a uniquely defined pressure.

It is clear that this problem has a unique solution $(\overline{\mathbf{w}}, \pi) \in \mathbf{H}_{s+2} \times H^{s+1}(\mathbb{T}_3)$, for any $\mathbf{w} \in \mathbf{H}_s$. We put $G(\mathbf{w}) = \overline{\mathbf{w}}$, $A = G^{-1}$. Notice that even if we work with real valued fields, $G = A^{-1}$ maps more generally $\mathbb{H}_s$ onto $\mathbb{H}_{s+2}$. Observe also that -in terms of Fourier series- when one inserts in (2.2)

$$\mathbf{w} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{w}}_{\mathbf{k}} \, e^{+i\mathbf{k}\cdot\mathbf{x}},$$

one easily gets, by searching $(\overline{\mathbf{w}}, \pi)$ in terms of Fourier Series, that

$$\overline{\mathbf{w}}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \frac{1}{1+\alpha^2|\mathbf{k}|^2} \widehat{\mathbf{w}}_{\mathbf{k}} \, e^{+i\mathbf{k}\cdot\mathbf{x}} = G(\mathbf{w}), \qquad \text{and} \qquad \pi = 0. \tag{2.3}$$

With a slight abuse of notation, for a scalar function χ we still denote by $\overline{\chi}$ the solution of the pure Helmholtz problem

$$A\overline{\chi} = -\alpha^2 \Delta \overline{\chi} + \overline{\chi} = \chi \quad \text{in } \mathbb{T}_3, \qquad G(\chi) = \overline{\chi}. \tag{2.4}$$

and of course there are not vanishing-mean conditions to be imposed for such cases. This notation –which is nevertheless historical– is motivated from the fact that in the periodic setting and for divergence-free vector fields the Stokes filter (2.2) is exactly the same as (2.4). Observe in particular that in the LES model (1.3) and in the filtered equations (1.2)-(1.4), the symbol "—" denotes the pure Helmholtz filter, applied component-by-component to the tensor fields $D_N(\mathbf{w}) \otimes D_N(\mathbf{w})$, $\mathbf{u} \otimes \mathbf{u}$, and $A\mathbf{w} \otimes A\mathbf{w}$ respectively.

2.4 The deconvolution operator

We start this section with a useful definition, which we shall use several times in the sequel to understand the relevant properties of the LES model.

Definition 2.1. *Let K be an operator acting on $\mathbf{H}_s$. Assume that $e^{-i\mathbf{k}\cdot\mathbf{x}}$ are eigen-vectors of K with corresponding eigenvalues $\widehat{K}_{\mathbf{k}}$. Then we shall say that $\widehat{K}_{\mathbf{k}}$ is the symbol of K.*

The deconvolution operator D_N is constructed thanks to the Van-Cittert algorithm, and is formally defined by

$$D_N := \sum_{n=0}^{N} (\mathrm{I} - G)^n. \tag{2.5}$$

The reader will find a complete description and analysis of the Van-Cittert Algorithm and its variants in [18]. Here we just report the properties we only need for the description of the model.
Starting from (2.5), we can express the deconvolution operator in terms of Fourier Series by the formula

$$D_N(\mathbf{w}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{D}_N(\mathbf{k})\widehat{\mathbf{w}}_{\mathbf{k}} e^{+i\mathbf{k}\cdot\mathbf{x}},$$

where

$$\widehat{D}_N(\mathbf{k}) = \sum_{n=0}^{N} \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^n = (1+\alpha^2|\mathbf{k}|^2)\rho_{N,\mathbf{k}}, \quad \rho_{N,\mathbf{k}} = 1 - \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^{N+1}. \tag{2.6}$$

The symbol $\widehat{D}_N(\mathbf{k})$ of the operator D_N satisfies the following

$$\text{for each } \mathbf{k}\in\mathcal{T}_3 \text{ fixed} \qquad \widehat{D}_N(\mathbf{k}) \to 1+\alpha^2|\mathbf{k}|^2, \quad \text{as } N\to+\infty, \tag{2.7}$$

even if not uniformly in $\mathbf{k}$. This means that $\{D_N\}_{N\in\mathbb{N}}$ converges to A in some sense (see Lemma 2.2 below). We need to specify this convergence in order to pass to the limit more than in "a formal way," to go from (1.3) to (1.4). A general goal for the all paper is to precisely determine the notion of $D_N \to A$ and to obtain enough estimates on the solution $\mathbf{w}$ of (1.3) in order to perform such limit.
The basic properties satisfied by $\widehat{D}_N$ that we will need are summarized in the following lemma.

Lemma 2.1. *For each $N\in\mathbb{N}$ the operator $D_N : \mathbf{H}_s \to \mathbf{H}_s$ is self-adjoint, it commutes with differentiation, and the following properties hold true:*

$$1 \le \widehat{D}_N(\mathbf{k}) \le N+1 \qquad \forall\,\mathbf{k}\in\mathcal{T}_3, \tag{2.8}$$

$$\widehat{D}_N(\mathbf{k}) \approx (N+1)\frac{1+\alpha^2|\mathbf{k}|^2}{\alpha^2|\mathbf{k}|^2} \qquad \textit{for large } |\mathbf{k}|, \tag{2.9}$$

$$\lim_{|\mathbf{k}|\to+\infty} \widehat{D}_N(\mathbf{k}) = N+1 \qquad \textit{for fixed } \alpha>0, \tag{2.10}$$

$$\widehat{D}_N(\mathbf{k}) \le (1+\alpha^2|\mathbf{k}|^2) \qquad \forall\,\mathbf{k}\in\mathcal{T}_3,\ \alpha>0. \tag{2.11}$$

All these claims are "obvious," in the sense that they follow from direct inspection of the formula (2.6). Nevertheless, they call for some comments. A first observation is that (2.10) is a direct consequence of (2.9). This says that the $\mathbf{H}_s$ spaces are preserved by the operator D_N. More precisely, for all $s\ge 0$, the map

$$\mathbf{w} \mapsto D_N(\mathbf{w}),$$

is an isomorphism which satisfies

$$\|D_N\|_{\mathbf{H}_s} = O(N+1).$$

Moreover, the term $\overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})}$ in model (1.3) is better than the convective term $\overline{A\mathbf{w} \otimes A\mathbf{w}}$ in the classical filtered Navier-Stokes Equations, making a good hope for the model to have what we call a unique "regular weak" solution (see Definition 3.1 in the next section) which satisfies an energy equality. This follows because the sequence $\{D_N\}_{N\in\mathbb{N}}$ is made of differential operators of zero-order approximating A, which is of the second-order. The solutions of (1.3) are stronger than the usual weak (dissipative) Leray's solution: this is the good news. The bad news is that high frequency modes are not under control and may blow up when one lets N to go to infinity, making very hard the question of the limit behavior of the sequence of models (1.3), when N goes to infinity.
In the same spirit of limiting behavior of D_N –as a byproduct of (2.7) and (2.9)– it can be shown that the sequence of operators $\{D_N\}_{N\in\mathbb{N}}$ "weakly" converges (more precisely one has point-wise convergence) to the operator A. The following result holds true.

Lemma 2.2. *Let $s \in \mathbb{R}$ and let $\mathbf{w} \in \mathbf{H}_{s+2}$. Then*

$$\lim_{N\to+\infty} D_N(\mathbf{w}) = A\mathbf{w} \qquad \textit{in } \mathbf{H}_s.$$

The proof of this lemma is very close to the one of Lemma 2.5 in [15]. Therefore, we skip the details.

Remark 2.1. *Since D_N is self-adjoint and non-negative it is possible to define the fractional powers D_N^α, for $\alpha \geq 0$. From the previous result we also obtain directly that if $\mathbf{w} \in \mathbf{H}_s$, then*

$$\forall\, \alpha \geq 0, \qquad \lim_{N\to+\infty} A^{-\alpha} D_N^\alpha(\mathbf{w}) = \mathbf{w} \qquad \textit{in } \mathbf{H}_s.$$

Remark 2.2. *The reader may observe that most of the properties satisfied by D_N are also satisfied by the Yosida approximation*

$$A_\lambda := \frac{\mathrm{I} - (\mathrm{I} + \lambda A)^{-1}}{\lambda}, \qquad \lambda > 0,$$

which is very common in the theory of semi-groups or in the calculus of variations. To compare the behavior of the two approximations, we write the explicit expression for the symbol of the Yosida approximation, with $\lambda = 1/N$:

$$\widehat{A}_{1/N}(\mathbf{k}) = (1+\alpha^2|\mathbf{k}|^2)\left[1 - \frac{1+\alpha^2|\mathbf{k}|^2}{N+1+\alpha^2|\mathbf{k}|^2}\right] = (1+\alpha^2|\mathbf{k}|^2)\sigma_{N,\mathbf{k}}, \quad \sigma_{N,\mathbf{k}} = \frac{N}{N+1+\alpha^2|\mathbf{k}|^2}.$$

One can directly compute that the asymptotics are essentially the same as in Lemma 2.1, but the Van Cittert operator D_N converges to A much faster than the Yosida approximation $A_{1/N}$, as N goes to infinity.

3 Existence results

As we pointed out in the introduction, in this paper we consider the initial value problem for the LES model (1.3). The aim of this section is to prove the existence of a unique solution to the system (1.3) for a given N. In the whole paper, $\alpha > 0$ is fixed as we already have said, and we assume that the data are such that

$$\mathbf{u}_0 \in \mathbf{H}_0, \quad \mathbf{f} \in L^2([0,T]; \mathbf{H}_{-1}), \tag{3.1}$$

which naturally yields

$$\overline{\mathbf{u}_0} \in \mathbf{H}_2, \quad \overline{\mathbf{f}} \in L^2([0,T]; \mathbf{H}_1). \tag{3.2}$$

We start by defining the notion of what we call a "regular weak" solution to this system.

Definition 3.1 ("Regular weak" solution)**.** *We say that the couple* $(\mathbf{w}, q)$ *is a "regular weak" solution to system (1.3) if and only if the three following items are satisfied:*

1) Regularity

$$\mathbf{w} \in L^2([0,T]; \mathbf{H}_2) \cap C([0,T]; \mathbf{H}_1), \tag{3.3}$$
$$\partial_t \mathbf{w} \in L^2([0,T]; \mathbf{H}_0) \tag{3.4}$$
$$q \in L^2([0,T]; H^1(\mathbb{T}_3)), \tag{3.5}$$

2) Initial data

$$\lim_{t \to 0} \|\mathbf{w}(t,\cdot) - \overline{\mathbf{u}_0}\|_{\mathbf{H}_1} = 0, \tag{3.6}$$

3) Weak Formulation

$$\forall\, \mathbf{v} \in L^2([0,T]; H^1(\mathbb{T}_3)^3), \tag{3.7}$$
$$\int_0^T \int_{\mathbb{T}_3} \partial_t \mathbf{w} \cdot \mathbf{v} - \int_0^T \int_{\mathbb{T}_3} \overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})} : \nabla \mathbf{v} + \nu \int_0^T \int_{\mathbb{T}_3} \nabla \mathbf{w} : \nabla \mathbf{v} + \int_0^T \int_{\mathbb{T}_3} \nabla q \cdot \mathbf{v} = \int_0^T \int_{\mathbb{T}_3} \overline{\mathbf{f}} \cdot \mathbf{v}. \tag{3.8}$$

Almost all terms in (3.8) are obviously well-defined thanks to (3.3)-(3.4)-(3.5), together with (3.7). The convective term however, needs to be checked a little bit more carefully. To this end, we first recall that D_N maps $\mathbf{H}_s$ onto itself, and the Sobolev embedding implies that $\mathbf{w} \in C([0,T]; \mathbf{H}_1) \subset L^\infty([0,T]; L^6(\mathbb{T}_3)^3)$.
Then, we still have $D_N(\mathbf{w}) \in C([0,T]; \mathbf{H}_1) \subset L^\infty([0,T]; L^6(\mathbb{T}_3)^3)$. In particular,

$$D_N(\mathbf{w}) \otimes D_N(\mathbf{w}) \in L^\infty([0,T]; L^3(\mathbb{T}_3)^3)^2.$$

Consequently, we have at least

$$\overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})} \in L^\infty([0,T]; H^2(\mathbb{T}_3)^3)^2 \subset L^\infty([0,T] \times \mathbb{T}_3)^9,$$

which yields the integrability of $\overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})} : \nabla \mathbf{v}$ for any $\mathbf{v} \in L^2([0,T]; H^1(\mathbb{T}_3)^3)$.

Remark 3.1. *We point out that we use the name "regular weak" solution, since* $(\mathbf{w}, q)$ *is a solution in the sense of distributions, but it will turn out to be smooth enough to be uniqueness and to satisfy an energy equality in place of only an energy inequality, such as in the usual Navier-Stokes equations.*

Theorem 3.1. *Assume that (3.1) holds,* $\alpha > 0$ *and* $N \in \mathbb{N}$ *are given and fixed. Then Problem* (1.3) *has a unique regular weak solution.*

Proof. We use the usual Galerkin method (see for instance the basics for the Navier-Stokes Equations in [20]). This allows to construct the velocity part of the solution, since the equation is projected on a divergence-free vector field space. The pressure is recovered by De Rham Theorem at the end of the process, that we divide into five steps:

STEP 1: we start by constructing approximate solutions $\mathbf{w}_m$, solving differential equations on finite dimensional spaces (see Definition 3.9 below);

STEP 2: we look for bounds on $\{\mathbf{w}_m\}_{m\in\mathbb{N}}$ and $\{\partial_t \mathbf{w}_m\}_{m\in\mathbb{N}}$, uniform with respect to $m \in \mathbb{N}$, in suitable spaces. This is obtained by using an energy-type equality satisfied by $A^{1/2}D_N^{1/2}(\mathbf{w}_m)$. Most of these bounds will result also *uniform in* N, where $N \in \mathbb{N}$ is the index related to the order of deconvolution of the model;

STEP 3: we use the main rules of functional analysis to get compactness properties about the sequence $\{\mathbf{w}_m\}_{n\in\mathbb{N}}$. This will allow us to pass to the limit when $m \to \infty$ and N is fixed, to obtain a solution to the model;

STEP 4: we check the question of the initial data;

STEP 5: we show that the solution we constructed is unique thanks Gronwall's lemma.

Since Step 1 and 3 are very classical, we will only sketch them, as well as Step 4 which is very close from what has already been done in [5, 19, 24]. On the other hand, Step 2 is one of the main original contributions in the paper and will also be useful in the next section. Indeed, we obtain many estimates, uniform in N, that allow us in passing to the limit when N goes to infinity and proving Theorem 1.1. Also Step 4 needs some application of classical tools in a way that is less standard than usual. We also point out that Theorem 3.1 greatly improves the corresponding existence result in [7] and it is not a simple restatement of those results.

STEP 1 : CONSTRUCTION OF VELOCITY'S APPROXIMATIONS.

Let be given $m \in \mathbb{N}^\star$ and define $\mathbf{V}_m$ to be the space of real valued trigonometric polynomial vector fields of degree less or equal than n, with vanishing both divergence and mean value on the torus $\mathbb{T}_3$,

$$\mathbf{V}_m := \{\mathbf{w} \in \mathbf{H}_1 : \quad \int_{\mathbb{T}_3} \mathbf{w}(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}} = \mathbf{0}, \quad \forall\, \mathbf{k}, \text{ with } |\mathbf{k}| > m\}. \tag{3.9}$$

The space $\mathbf{V}_m$ has finite dimension, denoted by d_m. Moreover, $\mathbf{V}_m \subset \mathbf{V}_{m+1}$ and, in the meaning of Hilbert spaces, $\mathbf{H}_1 = \cup_{m\in\mathbb{N}^\star}\mathbf{V}_m$. We notice that $\mathbf{V}_m$ is a subset of the finite dimensional space

$$\mathbf{W}_m := \{\mathbf{w} : \mathbb{T}_3 \to \mathbb{C}^3,\ \mathbf{w} = \sum_{\mathbf{k}\in\mathcal{T}_3, |\mathbf{k}|\le m} \widehat{\mathbf{w}}_{\mathbf{k}}\, e^{+i\mathbf{k}\cdot\mathbf{x}}\},$$

and the space $\mathbf{V}_m$ can be described as

$$\mathbf{V}_m := \mathbf{W}_m \cap \mathbf{H}_0. \tag{3.10}$$

We denote by $(\mathbf{e}_1, \dots, \mathbf{e}_{d_m})$ an orthogonal basis of $\mathbf{V}_m$. Remark that this basis is not made of the $e^{+i\mathbf{k}\cdot\mathbf{x}}$'s. Nevertheless, we do not need to know it precisely. Moreover, the family $\{\mathbf{e}_j\}_{j\in\mathbb{N}}$ is an orthogonal basis of $\mathbf{H}_0$ as well as of $\mathbf{H}_1$. As we shall see in the following, the $\mathbf{e}_j$'s can be chosen to be eigen-vectors of A, with $\|\mathbf{e}_j\| = 1$.
Let $\mathbb{P}_m$ denote the orthogonal projection from $\mathbf{H}_s$ $(s = 0, 1)$ onto $\mathbf{V}_m$. For instance, for $\mathbf{w}_0 = \overline{\mathbf{u}_0} = \sum_{j=1}^\infty w_j^0 \mathbf{e}_j$, we have

$$\mathbb{P}_m(\overline{\mathbf{u}_0}) = \sum_{j=1}^{d_m} w_j^0 \mathbf{e}_j.$$

In order to use classical tools for systems of ordinary differential equations, we approximate the external force by means of a standard Friederichs mollifier, see e.g. [23, 24]. Let ρ be an even function such that $\rho \in C_0^\infty(\mathbb{R})$, $0 \leq \rho(s) \leq 1$, $\rho(s) = 0$ for $|s| \geq 1$, and $\int_\mathbb{R} \rho(s)\, ds = 1$. Then, set $\mathbf{F}(t) = \overline{\mathbf{f}}(t)$ if $t \in [0, T]$ and zero elsewhere and for all positive ϵ define $\overline{\mathbf{f}}_\epsilon$, the smooth (with respect to time) approximation of $\overline{\mathbf{f}}$, by

$$\overline{\mathbf{f}}_\epsilon(t) := \frac{1}{\epsilon} \int_\mathbb{R} \rho\left(\frac{t-s}{\epsilon}\right) \mathbf{F}(s)\, ds.$$

Well known results imply that if (3.1) is satisfied, then $\overline{\mathbf{f}}_\epsilon \to \overline{\mathbf{f}}$ in $L^2([0, T]; \mathbf{H}_1)$. Thanks to Cauchy-Lipschitz Theorem, we know the existence of a unique function

$$\mathbf{w}_m(t, \mathbf{x}) = \sum_{j=1}^{d_m} w_{m,j}(t)\, \mathbf{e}_j(\mathbf{x})$$

and of a positive T_m such that the vector $(w_{m,1}(t), \ldots, w_{m,d_m}(t))$ is a C^1 solution on $[0, T_m] \subseteq [0, T]$, with $w_{m,j}(0) = w^0_{m,j}$, in the sense that $\forall\, \mathbf{v} \in \mathbf{V}_m$, $\forall\, t \in [0, T_m]$ it holds

(3.11)
$$\int_{\mathbb{T}_3} \partial_t \mathbf{w}_m(t, \mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x} - \int_{\mathbb{T}_3} (\overline{D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m)})(t, \mathbf{x}) : \nabla \mathbf{v}(\mathbf{x})\, d\mathbf{x}$$
$$+\nu \int_{\mathbb{T}_3} \nabla \mathbf{w}_m(t, \mathbf{x}) : \nabla \mathbf{v}(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{T}_3} \overline{\mathbf{f}}_{1/m}(t, \mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x},$$

where

$$\partial_t \mathbf{w}_m = \sum_{j=1}^{d_m} \frac{d w_{m,j}(t)}{dt}\, \mathbf{e}_j.$$

As we shall see it in step 2, we can take $T_m = T$. This ends the local-in-time construction of the approximate solutions $\mathbf{w}_m(t, \mathbf{x})$. □

Remark 3.2. *We want to stress to the reader's attention that a more precise notation would be*

$$\mathbf{w}_{m,N,\alpha},$$

instead of $\mathbf{w}_m$. *We are asking for this simplification to avoid a too heavy notation, since in this section both* N *and* α *are fixed.*

STEP 2. ESTIMATES.

As in the classical Galerkin method, we need some *a priori* estimates, first to show that the solution of the $(d_m \times d_m)$-systems of ordinary differential equations satisfied by $w_{m,j}$ exists in some non-vanishing time-interval, not depending on m (to this end a energy-type estimate is enough). Next, we want to obtain estimates on the $\mathbf{w}_m$'s and the $\partial_t \mathbf{w}_m$ for compactness properties, to pass to the limit, when $m \to \infty$ and N is still kept fixed.

As usual, we need to identify suitable test vector fields in (3.11) such that, the scalar product with the nonlinear term vanishes (if such choice does exist). We observe that the natural candidate is $AD_N(\mathbf{w}_m)$. Indeed, since A is self-adjoint and commutes with differential operators, it holds:

$$\int_{\mathbb{T}_3} (\overline{D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m)}) : \nabla(AD_N(\mathbf{w}_m))\, d\mathbf{x}$$
$$= \int_{\mathbb{T}_3} G\big(D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m)\big) : \nabla(AD_N(\mathbf{w}_m))\, d\mathbf{x}$$
$$= \int_{\mathbb{T}_3} (AG)\big(D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m)\big) : \nabla(D_N(\mathbf{w}_m))\, d\mathbf{x} = 0,$$

because $A \circ G = \mathrm{Id}$ on $\mathbf{H}_s$, $\nabla \cdot \big(D_N(\mathbf{w}_m)\big) = 0$, and thanks to the periodicity. This yields the equality

$$(3.12) \qquad \big(\partial_t \mathbf{w}_m, AD_N(\mathbf{w}_m)\big) - \nu\big(\Delta \mathbf{w}_m, AD_N(\mathbf{w}_m)\big) = \big(\overline{\mathbf{f}}_{1/m}, AD_N(\mathbf{w}_m)\big).$$

This formal computation asks for two clarifications:

i) We must check that $AD_N(\mathbf{w}_m)$ is a "legal" test function, to justify the above formal procedure. This means that for any fixed time t, we must prove that $AD_N(\mathbf{w}_m) \in \mathbf{V}_m$.

ii) Estimate (3.12) does not give a direct information about $\mathbf{w}_m$ itself and/or $\partial_t \mathbf{w}_m$. Therefore one must find how to deduce suitable estimates from it.

Point i) is the most simple to handle. On one hand, we already know that $G(\mathbf{H}_0) = \mathbf{H}_2 \subset \mathbf{H}_0$. On the other hand, formula (2.3) makes sure that $G(\mathbf{W}_m) \subset \mathbf{W}_m$. We now use representation (3.10), and we deduce that $G(\mathbf{V}_m) \subset \mathbf{V}_m$. Finally, it is clear that $\mathrm{Ker}(G) = \mathbf{0}$. Therefore, since $\mathbf{V}_m$ has a finite dimension, we deduce that G is an isomorphism on it. Then the space $\mathbf{V}_m$ is stable under the action of the operator A as well as under that of D_N. This makes $AD_N(\mathbf{w}_m)(t, \cdot) \in \mathbf{V}_m$ a "legal" multiplier in formulation (3.11), for each fixed t. Moreover, since A and D_N are self-adjoint operator that commute, one can choose the basis $(\mathbf{e}_1, \cdots, \mathbf{e}_{d_m}, \cdots)$ such that each $\mathbf{e}_j$ is still an eigen-vector of the operator A and D_N together. Therefore, the projection $\mathbb{P}_m$ commutes with A as well as with all by-products of A, such as D_N for instance. We shall use this remark later in the estimates.

The next point ii) is not so direct and constitutes the heart of the matter of this paper. The key observation is that the following identities hold:

$$(3.13) \qquad \big(\partial_t \mathbf{w}_m, AD_N(\mathbf{w}_m)\big) = \frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\|^2,$$

$$(3.14) \qquad \big(-\Delta \mathbf{w}_m, AD_N(\mathbf{w}_m)\big) = \|\nabla A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\|^2,$$

$$(3.15) \qquad \big(\overline{\mathbf{f}}_{1/m}, AD_N(\mathbf{w}_m)\big) = \big(A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\overline{\mathbf{f}}_{1/m}), A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\big).$$

These equalities are straightforward because A and D_N both commute, as well as they do with all differential operators. Therefore (3.12) or better

$$\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\|^2 + \|\nabla A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\|^2 = (A^{\frac{1}{2}} D_N^{\frac{1}{2}}\big(\overline{\mathbf{f}}_{1/m}), A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\big)$$

shows that the natural quantity under control is $A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)$. As we shall see in the remainder, norms of this quantity do control $\mathbf{w}_m$, as well as the natural key variable $D_N(\mathbf{w}_m)$. Finally, this yields an estimate for $\partial_t \mathbf{w}_m$.

Since we need to prove many *a priori* estimates, for the reader's convenience we organize the results in the following Table (3.16). We hope that having a bunch of estimates collected together will help in understanding the result. In a first reading one can skip the proof of the inequalities, in order to get directly into the core of the main result.
The results are organised as follows: In the first column we have labeled the estimates. The second column precises the variable under concern. The third one explains the bound in term of space function. The title of the space means that the considered sequence is bounded in this space. To be more precise, $E_{m,N} \in F$ for any variable $E_{m,N}$ and a space

F, means that the sequence $\{E_{m,N}\}_{(m,N)\in\mathbb{N}^2}$ is bounded in the space F. Finally the fourth column precises the order in terms of α, m, and N for each bound. Of course each bound is of order of magnitude

$$O\Big(\|\mathbf{u}_0\|_{L^2} + \frac{1}{\nu}\|\mathbf{f}\|_{L^2([0,T];L^2)}\Big),$$

and this why we do not mention it in the table. We also stress that all bounds are uniform in m. All bounds except (3.16-h) are uniform also in N. Moreover, we shall also see that they are uniform in T yielding, $T_m = T$ for each T. We mention that we could take $T=\infty$ at this level of our analysis, so far the source term $\mathbf{f}$ is defined on $[0,\infty[$.

(3.16)

Label	Variable	bound	order
$a)$	$A^{\frac12}D_N^{\frac12}(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$b)$	$D_N^{1/2}(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$c)$	$D_N^{1/2}(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_1)\cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
$d)$	$\mathbf{w}_m$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$e)$	$\mathbf{w}_m$	$L^\infty([0,T];\mathbf{H}_1)\cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
$f)$	$D_N(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
$g)$	$D_N(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_1)\cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1}+(N+1))$
$h)$	$\partial_t\mathbf{w}_m$	$L^2([0,T];\mathbf{H}_0)$	$O(\alpha^{-1})$

Checking (3.16-a) — For the simplicity, we shall assume that $\mathbf{f}\in L^2([0,T]\times\mathbb{T}_3)^3$, but the proof holds true also for $\mathbf{f}\in L^2([0,T];\mathbf{H}_{-1})$: one has to substitute in (3.15) the integral over $\mathbb{T}_3$ with the duality pairing $\langle\,.\,\rangle$ between $\mathbf{H}_1$ and $\mathbf{H}_{-1}$ and estimate in a standard way the quantity $\big\langle\overline{\mathbf{f}}_{1/m}, AD_N(\mathbf{w}_m)\big\rangle = \big\langle A^{\frac12}D_N^{\frac12}(\overline{\mathbf{f}}_{1/m}), A^{\frac12}D_N^{\frac12}(\mathbf{w}_m)\big\rangle$. When one integrates (3.12) with respect to time on the time interval $[0,t]$ for any time $t\le T_m$, (by using (3.13)-(3.14)-(3.15), and Cauchy-Schwartz inequality) one gets

(3.17)
$$\frac12\|A^{\frac12}D_N^{\frac12}(\mathbf{w}_m)(t,\cdot)\|^2+\nu\int_0^t\|\nabla A^{\frac12}D_N^{\frac12}(\mathbf{w}_m)\|^2\,d\tau \\ \le\frac12\|A^{\frac12}D_N^{\frac12}\mathbb{P}_m\overline{\mathbf{u}_0}\|^2+\int_0^t\|A^{\frac12}D_N^{\frac12}(\overline{\mathbf{f}}_{1/m})\|\cdot\|A^{\frac12}D_N^{\frac12}(\mathbf{w}_m)\|\,d\tau.$$

Notice that $A^{\frac12}D_N^{\frac12}(\overline{\mathbf{f}}_\epsilon)=A^{-\frac12}D_N^{\frac12}(\mathbf{f}_\epsilon)$. Since the operator $A^{-\frac12}D_N^{\frac12}$ has for symbol $\rho_{N,\mathbf{k}}^{1/2}\le1$, then $\|A^{\frac12}D_N^{\frac12}\overline{\mathbf{f}}_\epsilon\|\le C\|\mathbf{f}\|$ (cf. also Remark 2.1 and the properties of classical mollifiers). Since $\mathbb{P}_m$ commutes with A and D_N we also have

$$\|A^{\frac12}D_N^{\frac12}\mathbb{P}_m\overline{\mathbf{u}_0}\|=\|\mathbb{P}_mA^{\frac12}D_N^{\frac12}\overline{\mathbf{u}_0}\|\le\|A^{\frac12}D_N^{\frac12}\overline{\mathbf{u}_0}\|\le\|\mathbf{u}_0\|.$$

By using Poincaré's inequality combined with Young's inequality, and standard properties of mollifiers, one gets

$$\frac12\|A^{\frac12}D_N^{\frac12}(\mathbf{w}_m)(t,\cdot)\|^2+\frac{\nu}{2}\int_0^t\|\nabla A^{\frac12}D_N^{\frac12}(\mathbf{w}_m)\|^2\,d\tau\le C(\|\mathbf{u}_0\|,\,\|\mathbf{f}\|_{L^2([0,T];\mathbf{H}_{-1})}). \tag{3.18}$$

When one returns back to the definition of $\mathbf{w}_m$, one obtain (as a by product of (3.18) and also because the $\mathbf{e}_j$'s are eigen-vectors for A and D_N and therefore for $A^{\frac12}D_N^{\frac12}$) that

$$\sum_{j=1}^{d_m}\rho_{N,j}w_{m,j}(t)^2\le C(\|\mathbf{u}_0\|,\|\mathbf{f}\|_{L^2([0,T];\mathbf{H}_{-1})}),$$

making sure that one can take $T_m = T$, since no $\rho_{N,j}$ vanishes.

Checking (3.16-b)-(3.16-c) — Let $\mathbf{v} \in \mathbf{H}_2$. Then, with obvious notations one has

$$\|A^{\frac{1}{2}}\mathbf{v}\|^2 = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} (1+\alpha^2|\mathbf{k}|^2)|\widehat{\mathbf{v}}_{\mathbf{k}}|^2 = \|\mathbf{v}\|^2 + \alpha^2\|\nabla\mathbf{v}\|^2.$$

It suffices to apply this identity to $\mathbf{v} = D_N^{\frac{1}{2}}(\mathbf{w}_m)$ and to $\mathbf{v} = \partial_i D_N^{\frac{1}{2}}(\mathbf{w}_m)$ $(i = 1,2,3)$ in (3.17) to get the claimed result.

Checking (3.16-d)-(3.16-e) — This is a direct consequence of (3.16-b)-(3.16-c) combined with (2.8), that we also can understand as

$$\|\mathbf{w}\|_s \leq \|D_N(\mathbf{w})\|_s \leq (N+1)\|\mathbf{w}\|_s,$$

for general $\mathbf{w}$ and for any $s \geq 0$. This explains how important is to have a "lower bound" for the operator D_N, since estimates on $D_N^\alpha \mathbf{w}$, $\alpha > 0$, are in inherited by $\mathbf{w}$.

Checking (3.16-f) — The operator $A^{\frac{1}{2}} D_N^{\frac{1}{2}}$ has for symbol $(1+\alpha^2|\mathbf{k}|^2)\rho_{N,\mathbf{k}}^{1/2}$ while the one of D_N is $(1+\alpha^2|\mathbf{k}|^2)\rho_{N,\mathbf{k}}$. Since $0 \leq \rho_{N,\mathbf{k}} \leq 1$, then $\|D_N(\mathbf{w})\|_s \leq \|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w})\|_s$ for general $\mathbf{w}$ and for any $s \geq 0$. Therefore, the estimate (3.16-f) is still a consequence of (3.16-a).

Checking (3.16-g) — This follows directly from (3.16-e) together with (2.8). This also explains why the result depends on N, since we are using now the *upper bound* on the norm of the operator D_N, while in the previous estimates, we used directly the equation as well as the *lower bound* for the Van Cittert operator D_N.

Remark 3.3. *The fact that this estimate is valid for each N, but the bound may grow with N is the main source of difficulties in passing to the limit as $N \to +\infty$. Also the lack of this uniform bound requires some work to show that (certain sequences of) solutions to* (1.3) *converge to the average of a weak (dissipative) solution of the Navier-Stokes equations.*

Checking (3.16-h) — Let us take $\partial_t \mathbf{w}_m \in \mathbf{V}_m$ as test vector field in (3.11). We get

$$\|\partial_t \mathbf{w}_m\|^2 + \int_{\mathbb{T}_3} \mathbf{A}_{N,m} \cdot \partial_t \mathbf{w}_m + \frac{\nu}{2}\frac{d}{dt}\|\nabla \mathbf{w}_m\|^2 = \int_{\mathbb{T}_3} \overline{\mathbf{f}}_{1/m} \cdot \partial_t \mathbf{w}_m,$$

where

$$\mathbf{A}_{N,m} := \overline{\nabla \cdot \big(D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m)\big)}. \tag{3.19}$$

So far $\mathbf{w}_m(0,\cdot) = \mathbb{P}_m(\overline{\mathbf{u}_0}) \in \mathbf{H}_2$ and obviously $\|\mathbb{P}_m(\overline{\mathbf{u}_0})\|_2 \leq C\alpha^{-1}\|\mathbf{u}_0\|$, we only have to check that $\mathbf{A}_{N,m}$ is bounded in $L^2([0,T]\times\mathbb{T}_3)^3$. Thanks to (3.16-f), it is easy checked with classical interpolation inequalities, that $D_N(\mathbf{w}_m) \in L^4([0,T]; L^3(\mathbb{T}_3)^3)$. Therefore, $D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m) \in L^2([0,T]; L^{3/2}(\mathbb{T}_3)^9)$. Because the operator $(\nabla\cdot)\circ G$ makes to "gain one derivative," we deduce that $\mathbf{A}_{N,m} \in L^2([0,T]; W^{1,3/2}(\mathbb{T}_3)^3)$, which yields to $\mathbf{A}_{N,m} \in L^2([0,T]\times\mathbb{T}_3)^3$ since $W^{1,3/2}(\mathbb{T}_3) \subset L^3(\mathbb{T}_3) \subset L^2(\mathbb{T}_3)$ and $L^2([0,T]; L^2(\mathbb{T}_3)^3)$ is isomorphic to $L^2([0,T]\times\mathbb{T}_3)^3$ (see [19]). Moreover, the bound is of order $O(\alpha^{-1})$ as well, because of the norm of the operator $(\nabla\cdot)\circ G$ that we do not need to specify, but where α is involved. The remainder of the proof is a very classical trick, based on Gronwall's lemma (see for instance in [18] for a detailed report about this method). Notice that this bound on the nonlinear term is not optimal, but fits with our requirements. □

STEP 3 : PASSING TO THE LIMIT IN THE EQUATIONS WHEN $m \to \infty$, AND N IS FIXED.

Thanks to the bounds (3.16) and classical tricks, we can extract from the sequence $\{\mathbf{w}_m\}_{n\in\mathbb{N}}$ a sub-sequence converging to a $\mathbf{w} \in L^\infty([0,T];\mathbf{H}_1) \cap L^2([0,T];\mathbf{H}_2)$. Using Aubin-Lions Lemma (here one uses (3.16-h) and again classical tricks), one has:

$$\mathbf{w}_m \to \mathbf{w} \text{ weakly in } L^2([0,T];\mathbf{H}_2), \tag{3.20}$$

$$\mathbf{w}_m \to \mathbf{w} \text{ strongly in } L^p([0,T];\mathbf{H}_1), \quad \forall\, p < \infty, \tag{3.21}$$

$$\partial_t \mathbf{w}_m \to \partial_t \mathbf{w} \text{ weakly in } L^2([0,T];\mathbf{H}_0). \tag{3.22}$$

This already implies that $\mathbf{w}$ satisfies (3.3)-(3.4). From (3.21) and the continuity of D_N in $\mathbf{H}_s$ we get that $D_N(\mathbf{w}_m)$ converges strongly to $D_N(\mathbf{w})$ in $L^4([0,T]\times\mathbb{T}_3)$. Then, $D_N(\mathbf{w}_m)\otimes D_N(\mathbf{w}_m)$ converges strongly to $D_N(\mathbf{w})\otimes D_N(\mathbf{w})$ in $L^2([0,T]\times\mathbb{T}_3)$. This convergence of $\mathbf{w}$, together with the fact that $\overline{\mathbf{f}}_{1/m}$ converges strongly, implies that for all $\mathbf{v} \in L^2([0,T];\mathbf{H}_1)$

$$\begin{aligned}\int_0^T\int_{\mathbb{T}_3} \partial_t \mathbf{w}\cdot\mathbf{v}\,d\mathbf{x}\,d\tau - \int_0^T\int_{\mathbb{T}_3} \overline{D_N(\mathbf{w})\otimes D_N(\mathbf{w})} : \nabla\mathbf{v}\,d\mathbf{x}\,d\tau \\ +\nu\int_0^T\int_{\mathbb{T}_3}\nabla\mathbf{w}:\nabla\mathbf{v}\,d\mathbf{x}\,d\tau = \int_0^T\int_{\mathbb{T}_3}\overline{\mathbf{f}}\cdot\mathbf{v}\,d\mathbf{x}\,d\tau.\end{aligned} \tag{3.23}$$

Arguing similarly to [18], we easily get that $\mathbf{w}$ satisfies (3.6).

We are almost in order to introduce the pressure. Before doing this, let us make so comments about the variational formulation above. We decided to take test vector fields in $L^2([0,T];\mathbf{H}_1)$ to be in accordance with classical presentations. The regularity of $\mathbf{w}$ however yields $\overline{\nabla\cdot\big(D_N(\mathbf{w})\otimes D_N(\mathbf{w})\big)} \in L^2([0,T]\times\mathbb{T}_3)^3$ as well as $\Delta\mathbf{w} \in L^2([0,T]\times\mathbb{T}_3)^3$. Consequently, one can take vector test fields $\mathbf{v} \in L^2([0,T];\mathbf{H}_0)$ in formulation (3.23) that we can rephrased as: $\forall\,\mathbf{v} \in L^2([0,T];\mathbf{H}_0)$,

$$\int_0^T\int_{\mathbb{T}_3}(\partial_t\mathbf{w} + \mathbf{A}_N - \nu\Delta\mathbf{w} - \overline{\mathbf{f}})\cdot\mathbf{v}\,d\mathbf{x}\,d\tau = 0, \tag{3.24}$$

where for convenience, we have set

$$\mathbf{A}_N := \overline{\nabla\cdot\big(D_N(\mathbf{w})\otimes D_N(\mathbf{w})\big)}.$$

Therefore, for almost every $t \in [0,T]$,

$$\mathbb{F}(t,\cdot) = (\partial_t\mathbf{w} + \mathbf{A}_N - \nu\Delta\mathbf{u} - \overline{\mathbf{f}})(t,\cdot) \in L^2(\mathbb{T}_3)^3$$

is orthogonal to divergence-free vector fields in $L^2(\mathbb{T}_3)^3$ and De Rham's Theorem applies. Before going into technical details, let us first recall that among all available versions of this theorem, the most understandable is the one given by L. Tartar in [23], a work that has been later reproduced in many other papers. Notice also that there is a very elementary proof in the periodic case [19]. Now, from (3.24), we deduce that for each Lebesgue point t of $\mathbb{F}$, there is a scalar function $q(t,\cdot) \in H^1(\mathbb{T}_3)$, such that $\mathbb{F} = -\nabla q$, that one can rephrase as

$$\partial_t\mathbf{w} + \mathbf{A}_N - \nu\Delta\mathbf{w} + \nabla q = \overline{\mathbf{f}}. \tag{3.25}$$

Without loss of generality, one can assume that $\nabla\cdot\mathbf{f} = 0$. Therefore, taking the divergence of equation (3.25) yields

$$\Delta q = \nabla\cdot\mathbf{A}_N,$$

which also easily yields $q \in L^2([0,T]; H^1(\mathbb{T}_3))$, closing this part of the construction. $\square$

STEP 4 : About the initial data.

We must check that $\mathbf{w}(0,\cdot)$ can be defined and that $\mathbf{w}(0,\cdot) = \overline{\mathbf{u}_0}$, such as defined in (3.6). Thanks to the estimates above (mainly $\partial_t \mathbf{w} \in L^2([0,T]; \mathbf{H}_0)$, together with the regularity about $\mathbf{w}$ that we get) one obviously has $\mathbf{w} \in C([0,T]; \mathbf{H}_1)$. This allows us to define $\mathbf{w}(0,\cdot) \in \mathbf{H}_1$ and also to guarantee

$$\lim_{t\to 0^+} \|\mathbf{w}(t,\cdot) - \mathbf{w}(0,\cdot)\|_{\mathbf{H}_1} = 0.$$

It remains to identify $\mathbf{w}(0,\cdot)$. The construction displayed in Step 1 yields for $m \in \mathbb{N}$,

$$\mathbf{w}_m(t,\mathbf{x}) = \mathbb{P}_m(\overline{\mathbf{u}_0})(\mathbf{x}) + \int_0^t \partial_t \mathbf{w}_m(s,\mathbf{x})\, ds, \tag{3.26}$$

an identity which holds in $C^1([0,T]\times\Omega)$. Because of the weak convergence of $(\partial_t \mathbf{w}_m)_{m\in\mathbb{N}}$ to $\partial_t \mathbf{w}$ in $L^2([0,T]; \mathbf{H}_0)$ and thanks to usual properties of the projection's operator $\mathbb{P}_m$, one can easily pass to the limit in (3.26) in a weak sense in the space $L^2([0,T]; \mathbf{H}_0)$, to obtain

$$\mathbf{w}(t,\mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}) + \int_0^t \partial_t \mathbf{w}(s,\mathbf{x})\, ds.$$

Therefore, $\mathbf{w}(0,\mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x})$, closing the question about the initial data. $\square$

STEP 5: Uniqueness.

Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be two solutions corresponding to the same data $(\mathbf{u}_0, \mathbf{f})$ and let us define as usual $\mathbf{W} := \mathbf{w}_1 - \mathbf{w}_2$. We want to take $AD_N(\mathbf{W})$ as test function in the equation satisfied by $\mathbf{W}$, because we suspect that this is the natural multiplier for an energy equality. But before doing this, we must first check that this guy lives in $L^2([0,T]\times\mathbb{T}_3)^3$ to be sure that this is a "legal" multiplier. Notice that AD_N has for symbol $(1+\alpha^2|\mathbf{k}|^2)^2 \rho_{N,\mathbf{k}} \approx (N+1)(1+\alpha^2|\mathbf{k}|^2)^2/\alpha^2|\mathbf{k}|^2 \approx (N+1)\alpha^2|\mathbf{k}|^2$ for large $|\mathbf{k}|$. Therefore, for each fixed $N\in\mathbb{N}$, AD_N is like a Laplacian and makes us loose two derivatives in space. (Only in the limit $N\to+\infty$ we will loose four derivatives!) Fortunately, $\mathbf{W}\in L^2([0,T];\mathbf{H}_2)$ and therefore $AD_N(\mathbf{W}) \in L^2([0,T]\times\mathbb{T}_3)^3$, making this guy a good candidate to be the multiplier we need. After using tricks already introduced in this paper, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|A^{\frac12}D_N^{\frac12}(\mathbf{W})\|^2 + \nu\|\nabla A^{\frac12}D_N^{\frac12}(\mathbf{W})\|^2 & \\ \leq\ & |((D_N(\mathbf{W})\cdot\nabla)D_N(\mathbf{w}_2), D_N(\mathbf{W}))|, \\ \leq\ & \|D_N(\mathbf{W})\|^2_{L^4(\mathbb{T}_3)}\|\nabla D_N(\mathbf{w}_2)\|, \\ \leq\ & \|D_N(\mathbf{W})\|^{1/2}\|\nabla D_N(\mathbf{W})\|^{3/2}\|\nabla D_N(\mathbf{w}_2)\|, \end{aligned} \tag{3.27}$$

where the last line is obtained thanks the well-known "Ladyženskaya inequality" for interpolation of L^4 with L^2 and H^1, see [11, Ch. 1]. Starting from the last line of (3.27), we use the inequality $\|D_N(\mathbf{W})\| \leq \|A^{\frac12}D_N^{\frac12}(\mathbf{W})\|$ together with $\|D_N(\nabla\mathbf{W})\| \leq \|A^{\frac12}D_N^{\frac12}(\nabla\mathbf{W})\|$, the fact that D_N and ∇ commute, $\|D_N\| = (N+1)$, the bound of $\mathbf{w}_2$ in $L^\infty([0,T];\mathbf{H}_1)$, and Young's inequality. We obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|A^{\frac12}D_N^{\frac12}(\mathbf{W})\|^2 + \nu\|\nabla A^{\frac12}D_N^{\frac12}(\mathbf{W})\|^2 & \\ \leq \frac{27(N+1)^4 \sup_{t\geq 0}\|\nabla \mathbf{w}_2\|}{32\nu^3} & \|A^{\frac12}D_N^{\frac12}(\mathbf{W})\|^2 + \frac{\nu}{2}\|\nabla A^{\frac12}D_N^{\frac12}(\mathbf{W})\|^2. \end{aligned}$$

In particular, we get

$$\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}D_N^{\frac{1}{2}}(\mathbf{W})\|^2 \leq \frac{27(N+1)^4 \sup_{t\geq 0}\|\nabla \mathbf{w}_2\|}{32\nu^3}\,\|A^{\frac{1}{2}}D_N^{\frac{1}{2}}(\mathbf{W})\|^2.$$

We deduce from Gronwall's Lemma that $A^{\frac{1}{2}}D_N^{\frac{1}{2}}(\mathbf{W}) = \mathbf{0}$ because $A^{\frac{1}{2}}D_N^{\frac{1}{2}}(\mathbf{W})(0,\cdot) = \mathbf{0}$. To conclude that $\mathbf{W} = \mathbf{0}$, we must show that the kernel of the operator $A^{\frac{1}{2}}D_N^{\frac{1}{2}}$ is reduced to $\mathbf{0}$. This is trivial, since this operator has for symbol $(1+\alpha^2|\mathbf{k}|^2)\rho_{N,\mathbf{k}} \approx \alpha|\mathbf{k}|$ for large values of $\mathbf{k}$. This symbol never vanishes and the equivalence at infinity shows that this operator is of same order of $\alpha|\nabla|$. Therefore, it is an isomorphism that maps $\mathbf{H}_s$ onto $\mathbf{H}_{s-1}$. This concludes that the considered kernel is reduced to zero, proving uniqueness of the solution. □

Remark 3.4. *As we have seen, $AD_N(\mathbf{w})$ is a legitimate test function in Equation (3.25). When using computation rules already detailed in the paper, we get the following energy equality satisfied by $A^{1/2}D_N^{1/2}(\mathbf{w})$,*

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}D_N^{1/2}(\mathbf{w})\|^2 + \nu\|\nabla A^{1/2}D_N^{1/2}(\mathbf{w})\|^2 = \big(A^{-1/2}D_N^{1/2}(\mathbf{f}), A^{1/2}D_N^{1/2}(\mathbf{w})\big).$$

As we shall see in the sequel it seems that it is not possible to pass to the limit $N \to +\infty$ directly in this "energy equality" and some work to obtain an "energy inequality" is needed.

4 Passing to the limit when $N \to \infty$

The aim of this section is the proof of main result of the paper, namely Theorem 1.1. We now denote, for a given $N \in \mathbb{N}$ by $(\mathbf{w}_N, q_N)$, the "regular weak" solution to Problem 1.3. For the sake of completeness and to avoid possible confusion between the Galerkin index m and the deconvolution index N, we write again the system satisfied by $\mathbf{w}_N$

$$\begin{aligned} \partial_t \mathbf{w}_N + \nabla\cdot(\overline{D_N(\mathbf{w}_N)\otimes D_N(\mathbf{w}_N)}) - \nu\Delta\mathbf{w}_N + \nabla q_N = \overline{\mathbf{f}} \quad &\text{in} \quad [0,T]\times\mathbb{T}_3,\\ \nabla\cdot\mathbf{w}_N = 0 \quad &\text{in} \quad [0,T]\times\mathbb{T}_3,\\ \mathbf{w}_N(0,\mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}) \quad &\text{in} \quad \mathbb{T}_3. \end{aligned} \tag{4.1}$$

(More precisely, for all $N \in \mathbb{N}$ we set $\mathbf{w}_N = \lim_{m\to+\infty}\mathbf{w}_{m,N,\alpha}$), and the scale $\alpha > 0$ is fixed.) We aim to prove that the sequence $\{(\mathbf{w}_N, q_N)\}_{N\in\mathbb{N}}$ has a sub-sequence which converges to some $(\mathbf{w}, q)$ that is a solution to the averaged Navier-Stokes Equations (1.4). Recall that $(A\mathbf{w}, Aq)$ will be a distributional solution to the Navier-Stokes Equations. This result gives a undeniable theoretical support for the study of this ADM model.

We divide the proof into two steps:

1. We search additional estimates, uniform in N, to get compactness properties about the sequences $\{D_N(\mathbf{w}_N)\}_{N\in\mathbb{N}}$ and $\{\mathbf{w}_N\}_{N\in\mathbb{N}}$;

2. We prove strong enough convergence in order to pass to the limit in the equation (4.1).

Of course the challenge in this process is to pass to the limit in the nonlinear term $D_N(\mathbf{w}_N)\otimes D_N(\mathbf{w}_N)$. This is why we seek for an estimate about $\partial_t D_N(\mathbf{w}_N)$, knowing that we already have some estimate for $D_N(\mathbf{w}_N)$. This is how we get a compactness property satisfied by $D_N(\mathbf{w}_N)$, that we use for passing to the limit in the nonlinear term.

STEP 1 : ADDITIONAL ESTIMATES.

We quote in the following table the estimates we shall use for passing to the limit. The Table (4.2) is organized as the previous one (3.16). Recall that -for simplicity- $E_N \in F$ for any variable E_N and a space F, means that the sequence $\{E_N\}_{N\in\mathbb{N}}$ is bounded in the space F.

(4.2)

Label	Variable	bound	order
a	$\mathbf{w}_N$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
b	$\mathbf{w}_N$	$L^\infty([0,T];\mathbf{H}_1)\cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
c	$D_N(\mathbf{w}_N)$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
d	$\partial_t\mathbf{w}_N$	$L^2([0,T]\times\mathbb{T}_3)^3$	$O(\alpha^{-1})$
e	q_N	$L^2([0,T];H^1(\mathbb{T}_3))\cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3))$	$O(\alpha^{-1})$
f	$\partial_t D_N(\mathbf{w}_N)$	$L^{4/3}([0,T];\mathbf{H}_{-1})$	$O(1)$

Estimates (4.2-a), (4.2-b), (4.2-c), and (4.2-d) have already been obtained in the previous section. Therefore, we just have to check (4.2-e) and (4.2-f).

Checking (4.2-e) — As usual, to obtain regularity properties of the pressure (at least in the space-periodic case), we take the divergence of (3.25) obtaining

$$-\Delta q_N = \nabla\cdot\mathbf{A}_N - \nabla\cdot\overline{\mathbf{f}},$$

where we recall that

$$\mathbf{A}_N = \overline{\nabla\cdot(D_N(\mathbf{w}_N)\otimes D_N(\mathbf{w}_N))}.$$

Next, since $\mathbf{f}\in L^2([0,T]\times\mathbb{T}_3)^3$, then we get $\nabla\cdot\overline{\mathbf{f}}\in L^2([0,T];H^1(\mathbb{T}_3)^3)$. We now investigate the regularity of $\mathbf{A}_N$. We already know from the estimates proved in the previous section that $\mathbf{A}_N\in L^2([0,T]\times\mathbb{T}_3)^3$. This yields the first bound in $L^2([0,T];H^1(\mathbb{T}_3))$ for q_N.

We now seek the other estimate for q_N. Classical interpolation inequalities combined with (4.2-c) yield $D_N(\mathbf{w}_N)\in L^{10/3}([0,T]\times\mathbb{T}_3)$. Therefore, $\mathbf{A}_N\in L^{5/3}([0,T];W^{1,5/3}(\mathbb{T}_3))$. Consequently, we obtain

$$q_N\in L^2([0,T];H^1(\mathbb{T}_3))\cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3)).$$

Checking (4.2-f) — Let be given $\mathbf{v}\in L^4([0,T];\mathbf{H}_1)$. We use $D_N(\mathbf{v})\in L^4([0,T];\mathbf{H}_1)$ as test function in the equation satisfied by $(\mathbf{w}_N,q_N)$, that is now (by the results previously proved) a completely justified computation. We get, by using that $\partial_t\mathbf{w}\in L^2([0,T]\times\mathbb{T}_3)^3$ (as well as all other guys in the equation), D_N commutes with differential operators, G and D_N are self-adjoint, the pressure term cancels because $\nabla\cdot D_N(\mathbf{v})=0$, and classical integrations by parts

$$\begin{aligned}(\partial_t\mathbf{w}_N,D_N(\mathbf{v})) &= (\partial_t D_N(\mathbf{w}_N),\mathbf{v})\\ &= \nu(\Delta\mathbf{w}_N,D_N(\mathbf{v})) - (D_N(\mathbf{w}_N)\otimes D_N(\mathbf{w}_N),\overline{D_N(\nabla\mathbf{v})}) - (D_N(\overline{\mathbf{f}}),\mathbf{v}).\end{aligned}\tag{4.3}$$

We first observe that

$$|(\Delta\mathbf{w}_N,D_N(\mathbf{v}))| = |(\nabla D_N(\mathbf{w}_N),\nabla\mathbf{v})|\le C_1(t)\|\mathbf{v}\|_1,\tag{4.4}$$

and we use the $L^2([0,T];H^1(\mathbb{T}_3)^3)$ bound for $D_N(\mathbf{w}_N)$, to infer that the function $C_1(t)\in L^2([0,T])$, with a bound uniform with respect to $N\in\mathbb{N}$. Using $\|D_N(\overline{\mathbf{f}})\|\le\|\mathbf{f}\|$ already

proved in the previous section and Poincaré's inequality, we handle the term concerning the external forcing as follows:

$$\big|(D_N(\overline{\mathbf{f}}), \mathbf{v})\big| \le C\, \|\mathbf{f}\|\, \|\mathbf{v}\|_1, \tag{4.5}$$

C being the Poincaré's constant. Finally, from (4.2-c) and usual interpolation inequalities, we obtain that $D_N(\mathbf{w}_N)$ belongs to $L^{8/3}([0,T]; L^4(\mathbb{T}_3)^3)$, which yields

$$D_N(\mathbf{w}_N) \otimes D_N(\mathbf{w}_N) \in L^{4/3}([0,T]; L^2(\mathbb{T}_3)^9).$$

Therefore, when we combine the latter estimate with $\|\overline{D_N(\nabla \mathbf{v})}\| \le \|\nabla \mathbf{v}\|$, to get

$$\big|\big(D_N(\mathbf{w}_N) \otimes D_N(\mathbf{w}_N), \overline{D}_N(\nabla \mathbf{v})\big)\big| \le C_2(t)\|\mathbf{v}\|_1, \tag{4.6}$$

where $C_2(t) \in L^{4/3}([0,T])$ and it is uniform in $N \in \mathbb{N}$. The final result is a consequence of (4.3) combined with (4.4), (4.5), (4.6), and $C_1(t) + \|\mathbf{f}(t,\cdot)\| + C_2(t) \in L^{4/3}([0,T])$, uniformly with respect to $N \in \mathbb{N}$. □

Step 2 : Passing to the limit.

From the above estimates and classical rules of functional analysis, we can infer that there exist

$$\begin{aligned} &\mathbf{w} \in L^\infty([0,T]; \mathbf{H}_1) \cap L^2([0,T]; \mathbf{H}_2),\\ &\mathbf{z} \in L^\infty([0,T]; \mathbf{H}_0) \cap L^2([0,T]; \mathbf{H}_1),\\ &q \in L^2([0,T]; H^1(\mathbb{T}_3)) \cap L^{5/3}([0,T]; W^{2,5/3}(\mathbb{T}_3)),\end{aligned}$$

such that, up to sub-sequences,

$$\begin{aligned} &\mathbf{w}_N \longrightarrow \mathbf{w} && \begin{cases} \text{weakly in } L^2([0,T]; \mathbf{H}_2),\\ \text{weakly}* \text{ in } L^\infty([0,T]; \mathbf{H}_1),\\ \text{strongly in } L^p([0,T]; \mathbf{H}_1), \quad \forall\, p < \infty, \end{cases}\\ &\partial_t \mathbf{w}_N \longrightarrow \partial_t \mathbf{w} && \text{weakly in } L^2([0,T] \times \mathbb{T}_3),\\ &D_N(\mathbf{w}_N) \longrightarrow \mathbf{z} && \begin{cases} \text{weakly in } L^2([0,T]; \mathbf{H}_1),\\ \text{weakly}* \text{ in } L^\infty([0,T]; \mathbf{H}_0),\\ \text{strongly in } L^p([0,T] \times \mathbb{T}_3)^3, \ \forall\, p < 10/3, \end{cases}\\ &\partial_t D_N(\mathbf{w}_N) \longrightarrow \partial_t \mathbf{z} && \text{weakly in } L^{4/3}([0,T]; \mathbf{H}_{-1}),\\ &q_N \longrightarrow q && \text{weakly in } L^2([0,T]; H^1(\mathbb{T}_3)) \cap L^{5/3}([0,T]; W^{2,5/3}(\mathbb{T}_3)). \end{aligned} \tag{4.7}$$

Of course, we have

$$D_N \mathbf{w}_N \otimes D_N \mathbf{w}_N \longrightarrow \mathbf{z} \otimes \mathbf{z} \quad \text{strongly in } L^p([0,T] \times \mathbb{T}_3)^9, \quad \forall\, p < 5/3. \tag{4.8}$$

All other terms in the equation pass easily to the limit as well. Our proof will be complete as soon as we shall have checked that $\mathbf{z} = A\mathbf{w}$, thanks to (4.8). However, this is almost straightforward as we shall see, the hard job being already done by the proof of the various estimates.

Let us consider $\mathbf{v} \in L^2([0,T]; \mathbf{H}_2)$. We have $(D_N(\mathbf{w}_N), \mathbf{v}) = (\mathbf{w}_N, D_N(\mathbf{v}))$. We claim that $D_N(\mathbf{v}) \to A\mathbf{v}$ strongly in $L^2([0,T] \times \mathbb{T}_3)^3$. This will suffice to conclude the proof. Indeed,

if such a convergence result holds, we have using (4.7) and computational rules already quoted,

$$
\begin{array}{ccc}
\big(D_N(\mathbf{w}_N),\mathbf{v}\big) & = & \big(\mathbf{w}_N, D_N(\mathbf{v})\big) \\
\big\downarrow & & \big\downarrow \\
(\mathbf{z},\mathbf{v}) & & (\mathbf{w}, A\mathbf{v}) \\
\big\| & & \big\| \\
(\mathbf{z},\mathbf{v}) & = & (A\mathbf{w},\mathbf{v})
\end{array}
$$

yielding $\mathbf{z} = A\mathbf{w}$.
It remains to check the claim. We can write

$$
\mathbf{v} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{v}}_{\mathbf{k}}(t)\, e^{+i\mathbf{k}\cdot\mathbf{x}},
$$

and consequently (see in [19])

$$
\|\mathbf{v}\|^2_{L^2([0,T];\mathbf{H}_2)} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^4 \int_0^T |\widehat{\mathbf{v}}_{\mathbf{k}}(t)|^2 dt < \infty.
$$

Let $\varepsilon > 0$ being given. Then, there exists $0 < K = K(\mathbf{v}) \in \mathbb{N}$ such that

$$
\sum_{|\mathbf{k}|>K} |\mathbf{k}|^4 \int_0^T |\widehat{\mathbf{v}}_{\mathbf{k}}(t)|^2 dt < \frac{\varepsilon}{2}.
$$

Since $0 \le (1-\rho_{N,\mathbf{k}}) \le 1$, we have

$$
\begin{aligned}
\int_0^T \|(A-D_N)\mathbf{v}\|^2 &= \sum_{\mathbf{k}\in\mathcal{T}_3^\star} (1+\alpha^2|\mathbf{k}|^2)^2(1-\rho_{N,\mathbf{k}})^2 \int_0^T |\widehat{\mathbf{v}}_{\mathbf{k}}(t)|^2 dt, \\
&= \sum_{0<|\mathbf{k}|\le K} (1+\alpha^2|\mathbf{k}|^2)^2(1-\rho_{N,\mathbf{k}})^2 \int_0^T |\widehat{\mathbf{v}}_{\mathbf{k}}(t)|^2 dt \\
&\quad + \sum_{|\mathbf{k}|>K} (1+\alpha^2|\mathbf{k}|^2)^2(1-\rho_{N,\mathbf{k}})^2 \int_0^T |\widehat{\mathbf{v}}_{\mathbf{k}}(t)|^2 dt, \\
&< \sum_{0<|\mathbf{k}|\le K} (1+\alpha^2|\mathbf{k}|^2)^2(1-\rho_{N,\mathbf{k}})^2 \int_0^T |\widehat{\mathbf{v}}_{\mathbf{k}}(t)|^2 dt + \frac{\varepsilon}{2}.
\end{aligned}
$$

Observe that – for each given $\mathbf{k} \in \mathcal{T}_3^\star$ – we have $\rho_{N,\mathbf{k}} \to 1$ when $N \to \infty$. Therefore, there exists $N_0 \in \mathbb{N}$ (obviously depending on $\mathbf{v}$ and on K) such that for all $N > N_0$,

$$
\sum_{|\mathbf{k}|\le K} (1+\alpha^2|\mathbf{k}|^2)^2(1-\rho_{N,\mathbf{k}})^2 \int_0^T |\widehat{\mathbf{v}}_{\mathbf{k}}(t)|^2 dt < \frac{\varepsilon}{2}.
$$

We thus obtained that

$$
\forall\, \varepsilon > 0 \quad \exists\, N_0 = N_0(\mathbf{v}) \in \mathbb{N} : \quad \|(A-D_N)\mathbf{v}\|^2_{L^2([0,T];\mathbf{H}_2)} < \varepsilon, \quad \forall\, N > N_0,
$$

ending the proof. $\square$

Remark 4.1. *Let* $(\mathbf{u}_N, p_N) = (D_N(\mathbf{w}_N), D_N(q_N))$*, and define* $(\mathbf{u}, p) := (A\mathbf{w}, Aq)$*. Our proof also shows that the field* $(\mathbf{u}_N, p_N)$ *satisfies the equation*

$$
\begin{aligned}
\partial_t \mathbf{u}_N + (D_N \circ G)\left(\nabla \cdot (\mathbf{u}_N \otimes \mathbf{u}_N)\right) - \nu \Delta \mathbf{u}_N + \nabla p_N &= (D_N \circ G)(\mathbf{f}),\\
\nabla \cdot \mathbf{u}_N &= 0,\\
\mathbf{u}_N(0, \mathbf{x}) &= (D_N \circ G)(\mathbf{u}_0)(\mathbf{x}).
\end{aligned}
\tag{4.9}
$$

This equation is consistent with the convergence result, since $D_N \circ G \to \mathrm{Id}$*, and the proof contains the fact that* $(\mathbf{u}, p) = \lim_{N\to+\infty}(\mathbf{w}_N, q_N)$ *is at least a distributional solution of the Navier-Stokes Equations* (1.1)*. We also recall that the energy equality holds (see Remark 3.4), for the solution* $(\mathbf{w}_N, q_N)$ *of the ADM model* (4.1)*.*

We now prove that the solution $\mathbf{w}$ satisfies an "energy inequality." Observe that, from the previous estimates, we obtained (as a consequence of the lower bound on the operator D_N) that

$$
\begin{aligned}
&\mathbf{w}_N \in L^2([0,T]; \mathbf{H}_2) \quad \text{uniformly in } N\\
&\mathbf{w}_N \in L^\infty([0,T]; \mathbf{H}_2) \quad \text{NON uniformly in } N,
\end{aligned}
$$

hence obtaining an estimate for $\mathbf{w}$ in $L^\infty(0,T;\mathbf{H}_2)$ is not trivial at all since it does not derive directly from the various estimates collected in the tables.

Proposition 4.1. *Let be given* $\mathbf{u}_0 \in \mathbf{H}_0$ *and* $\mathbf{f} \in L^2([0,T];\mathbf{H}_0)$*, and let* $\{(\mathbf{w}_N, q_N)\}_{N\in\mathbb{N}}$ *be a (possibly relabelled) sequence of regular weak solutions converging to a weak solution* $(\mathbf{w}, q)$ *of the filtered Navier-Stokes equations. Then,* $\mathbf{w}$ *satisfies the energy inequality*

$$
\frac{1}{2}\frac{d}{dt}\|A\mathbf{w}\|^2 + \nu\|\nabla A\mathbf{w}\|^2 \leq (\mathbf{f}, A\mathbf{w}),
$$

in the sense of distributions, that is

$$
-\frac{1}{2}\int_0^T \|A\mathbf{w}(s)\|^2 \phi'(s)\,ds + \nu\int_0^T \|\nabla A\mathbf{w}(s)\|^2\,\phi(s)\,ds \leq \int_0^T (\mathbf{f}(s), A\mathbf{w}(s))\,\phi(s)\,ds,
$$

for all $\phi \in C_0^\infty(0,T)$ *such that* $\phi \geq 0$ *(From this one can derive the more familiar integral formulation* (4.12)*. See [5, 9, 24]). This implies that* $\mathbf{w}$ *is the average of a weak (in the sense of Leray-Hopf) or dissipative solution* $\mathbf{u}$ *of the Navier-Stokes equation* (1.1)*. In fact, the energy inequality can also be read also as*

$$
\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \nu\|\nabla A\mathbf{u}\|^2 \leq (\mathbf{f}, \mathbf{u}).
$$

If we assume less regularity on the external force, as for instance $\mathbf{f} \in L^2([0,T];\mathbf{H}_{-1})$*, the proof remains the same and we obtain the corresponding inequality*

$$
\frac{1}{2}\frac{d}{dt}\|A\mathbf{w}\|^2 + \nu\|\nabla A\mathbf{w}\|^2 \leq \langle \mathbf{f}, A\mathbf{w}\rangle.
$$

Proof. We proved that a (relabelled) sub-sequence $\{\mathbf{w}_N\}_{n\in\mathbb{N}}$ converges to a vector field $\mathbf{w}$, which is solution in the sense of distributions of the filtered Navier-Stokes equations. The final step is to prove that the solution $\mathbf{w}$ satisfies the "energy inequality." In fact observe that the *a priori* estimate we proved show that

$$
\mathbf{w} \in L^2([0,T];\mathbf{H}_2) \iff \mathbf{u} \in L^2([0,T];\mathbf{H}_0),
$$

and consequently the solution $\mathbf{u}$ belongs to $L^2(\mathbb{T}_3)$, for a.e. $t \in [0,T]$. We need now to show that the L^2-norm is bounded uniformly and that an energy balance holds. To this end we recall that the following energy equality (see Remark 3.4), is satisfied, for each $N \geq 0$, by the solutions of (4.1)

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)\|^2 + \nu\|\nabla A^{1/2}D_N^{1/2}(\mathbf{w}_N)\|^2 = \big(A^{-1/2}D_N^{1/2}(\mathbf{f}), A^{1/2}D_N^{1/2}(\mathbf{w}_N)\big)$$

(The previous inequality has to be intended in the sense of distributions.) What is needed is to pass to the limit as $N \to +\infty$. First, let us write the equality in integral form, since it will be most useful for our purposes: for all $t \in [0,T]$

$$\begin{aligned} &\frac{1}{2}\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(t)\|^2 + \nu\int_0^t \|\nabla A^{1/2}D_N^{1/2}(\mathbf{w}_N)(s)\|^2\,ds \\ &\qquad = \frac{1}{2}\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(0)\|^2 + \int_0^t \big(A^{-1/2}D_N^{1/2}(\mathbf{f}), A^{1/2}D_N^{1/2}(\mathbf{w}_N)\big)\,ds. \end{aligned} \tag{4.10}$$

Observe now that, by the estimates in (3.16-b) we have established the following convergence

$$\begin{aligned} A^{1/2}D_N^{1/2}(\mathbf{w}_N) &\to A^{1/2}A^{1/2}(\mathbf{w}) \quad \text{weakly in } L^2([0,T];\mathbf{H}_1), \\ A^{1/2}D_N^{1/2}(\mathbf{w}_N) &\to A^{1/2}A^{1/2}(\mathbf{w}) \quad \text{weakly}* \text{ in } L^\infty([0,T];\mathbf{H}_0). \end{aligned} \tag{4.11}$$

This follows since, we are working now on a sequence $\{\mathbf{w}_N\}_{N\in\mathbb{N}}$ which is convergent in all the spaces previously used. In particular, we have that $\{A^{1/2}D_N^{1/2}(\mathbf{w}_N)\}_{N\in\mathbb{N}}$ converges to "something," weakly in $L^2([0,T];\mathbf{H}_1)$, and also $\{A^{1/2}D_N^{1/2}(\mathbf{w}_N)\}_{N\in\mathbb{N}}$ has a $L^\infty([0,T];\mathbf{H}_0)$ weak$*$ limit. Finally, since we previously showed that

$$\begin{aligned} D_N(\mathbf{w}_N) &\to A\mathbf{w} \qquad \text{strongly in } L^p([0,T]\times\mathbb{T}_3), \qquad \forall\, p < 10/3, \\ D_N(\mathbf{w}_N) &\to A\mathbf{w} \qquad \text{weakly in } L^2([0,T];\mathbf{H}_0), \end{aligned}$$

the same arguments as before show also that

$$D_N^{1/2}(\mathbf{w}_N) \to A^{1/2}(\mathbf{w}) \qquad \text{weakly in } L^2([0,T];\mathbf{H}_1),$$

and the uniqueness of the weak limit proves (4.11).
Next, due to the assumptions on $\mathbf{f}$ we have

$$A^{-1/2}D_N^{1/2}\mathbf{f} \to \mathbf{f} \qquad \text{strongly in } L^2([0,T];\mathbf{H}_0).$$

In addition, since for all $N \in \mathbb{N}$, $\mathbf{w}_N(0) = \mathbf{w}(0) = \overline{\mathbf{u}}(0) \in \mathbf{H}_2$, and since the integral involving the external force $\mathbf{f}$ consists one term weakly converging, times another one strongly converging, it holds that

$$\begin{aligned} \lim_{N\to+\infty}\left[\frac{1}{2}\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(0)\|^2 + \int_0^t (A^{-1/2}D_N^{1/2}(\mathbf{f}), A^{1/2}D_N^{1/2}(\mathbf{w}_N))\,ds\right] \\ = \frac{1}{2}\|A\mathbf{w}(0)\|^2 + \int_0^t (\mathbf{f}, A\mathbf{w})\,ds. \end{aligned}$$

The previous limit implies that the left-hand side of (4.10) is bounded uniformly in $N \in \mathbb{N}$ since

$$
\begin{aligned}
\limsup_{N\to+\infty} & \left[\frac{1}{2}\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(t)\|^2 + \nu\int_0^t \|\nabla A^{1/2}D_N^{1/2}(\mathbf{w}_N)(s)\|^2\,ds\right] \\
&\leq \liminf_{N\to+\infty}\left[\frac{1}{2}\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(0)\|^2 + \int_0^t (A^{-1/2}D_N^{1/2}(\mathbf{f}), A^{1/2}D_N^{1/2}(\mathbf{w}_N))\,ds\right] \\
&= \lim_{N\to+\infty}\left[\frac{1}{2}\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(0)\|^2 + \int_0^t (A^{-1/2}D_N^{1/2}(\mathbf{f}), A^{1/2}D_N^{1/2}(\mathbf{w}_N))\,ds\right] \\
&= \frac{1}{2}\|A\mathbf{w}(0)\|^2 + \int_0^t (\mathbf{f}(s), A\mathbf{w}(s))\,ds.
\end{aligned}
$$

Next, we use the elementary inequality for the real valued sequences $\{a_N\}_{n\in\mathbb{N}}$ and $\{b_N\}_{n\in\mathbb{N}}$

$$
\limsup_{N\to+\infty} a_N + \liminf_{N\to+\infty} b_N \leq \limsup_{N\to+\infty}(a_N + b_N),
$$

with

$$
a_N := \frac{1}{2}\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(t)\|^2 \quad \text{and} \quad b_N = \nu\int_0^t \|\nabla A^{1/2}D_N^{1/2}(\mathbf{w}_N)(s)\|^2\,ds.
$$

(The inequality holds since we know in advance that the right-hand side is finite.) We infer that

$$
\begin{aligned}
\limsup_{N\to+\infty}\frac{1}{2}\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(t)\|^2 + \liminf_{N\to+\infty}\nu\int_0^t \|\nabla A^{1/2}D_N^{1/2}(\mathbf{w}_N)(s)\|^2\,ds& \\
\leq \frac{1}{2}\|A\mathbf{w}(0)\|^2 + \int_0^t (\mathbf{f}(s), A\mathbf{w}(s))\,ds&.
\end{aligned}
$$

By lower semi-continuity of the norm this implies that

$$
\int_0^t \|\nabla A\mathbf{w}(s)\|^2\,ds \leq \liminf_{N\to+\infty}\int_0^t \|\nabla A^{1/2}D_N^{1/2}\mathbf{w}_N(s)\|^2\,ds.
$$

On the other hand, since $D^{1/2}\mathbf{w}_N \to A^{1/2}\mathbf{w}$ weakly$*$ $L^\infty([0,T];\mathbf{H}_0)$. Again by identification of the weak limit we get

$$
\|A\mathbf{w}(t)\|^2 \leq \limsup_{N\to+\infty}\|A^{1/2}D_N^{1/2}\mathbf{w}_N(t)\|^2.
$$

By collecting all the estimates, we have finally proved that, for all $t \in [0,T]$,

$$
\frac{1}{2}\|A\mathbf{w}(t)\|^2 + \nu\int_0^t \|\nabla A\mathbf{w}(s)\|^2\,ds \leq \frac{1}{2}\|A\mathbf{w}(0)\|^2 + \int_0^t (\mathbf{f}(s), A\mathbf{w}(s))\,ds. \tag{4.12}
$$

This can be read as the standard energy inequality for $\mathbf{u} = A\mathbf{w}$.

$$
\frac{1}{2}\|\mathbf{u}(t)\|^2 + \nu\int_0^t \|\nabla\mathbf{u}(s)\|^2\,ds \leq \frac{1}{2}\|\mathbf{u}(0)\|^2 + \int_0^t (\mathbf{f}(s), \mathbf{u}(s))\,ds, \qquad \forall\, t \in [0,T].
$$

□

Remark 4.2. *With slightly more effort (the techniques are the usual ones as for the Navier-Stokes equations, see e.g., [5]) we can prove the same inequality, between t_0 and t, where t_0 is any Lebesgue point of the function $\|A\mathbf{w}(t)\|$: For a.e. $t_0 \in [0,T]$ it holds*

$$
\frac{1}{2}\|A\mathbf{w}(t)\|^2 + \nu\int_{t_0}^t \|\nabla A\mathbf{w}(s)\|^2\,ds \leq \frac{1}{2}\|A\mathbf{w}(t_0)\|^2 + \int_{t_0}^t (\mathbf{f}(s), A\mathbf{w}(s))\,ds, \quad \forall\, t \in [t_0,T].
$$

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